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# Quantum Chaos

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A quantum system which is allowed to interact with its boundary in a self-consistent way is shown to exhibit chaos. We conjecture that in general genuine wave chaos (decaying autocorrelation functions, exponential sensitivity of wavefunctions to initial wavefunction configurations) can be obtained whenever a wavefield is allowed to modify its confining boundaries in a self-consistent way. We suggest to test this conjecture in the acoustic regime.

**KEY WORDS:** Nonlinear dynamics; chaos; quantum chaos.

A new chapter in the quest for quantum chaos began when Casati *et al.*[1] reported their results on the quantum dynamics of the kicked rotor. Classically this system is strongly chaotic,[2] which leads to an unlimited diffusive increase of rotational energy. Therefore, quantum mechanics—intuitively perceived as being more unpredictable than classical mechanics—was expected to show an even stronger stochastic behavior. But instead of chaos and unlimited chaotic diffusion, it was found that the diffusive energy gain of the quantum kicked rotor stopped at some critical interaction time $t^*$. From then on, i.e., for times $t$ larger than $t^*$, the quantum dynamics was marked by pronounced recurrences (collapses and revivals) which occurred on a time scale much shorter than the classical Poincaré recurrence times. Later, the quantum freeze of classical diffusion was interpreted as a manifestation of Anderson localization by Fishman *et al.*[3] This "negative" result notwithstanding, many investigators continued the search for quantum instabilities and quantum chaos. Despite a concentrated effort, none of the indicators of classical chaos, such as exponential sensitivity of wavefunctions to initial conditions or decaying autocorrelation functions, have been detected in any of the quantum systems investigated

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so far. This "failure" is usually attributed to the linearity of quantum mechanics (superposition principle).

While it is relatively easy to see why bounded time-independent quantum systems with rigid boundaries cannot be chaotic (they are characterized by a discrete spectrum and normalizable wavefunctions), it is harder to establish a similar result for the case of periodically or quasi-periodically driven quantum systems. Although a complicated quantum time evolution of expectation values can sometimes be observed,[4] it has to be refuted as not chaotic in the long-time limit.[5] On the other hand, systems like the kicked rotor[1-3] or quantum billiards[6-8] are characterized by a rigid setup. This means that the boundary conditions of the system are chosen once and for all and the quantum dynamics is studied subject to these *fixed* boundary conditions.

In this paper we will show that a quantum system interacting self-consistently with a *mobile* boundary exhibits genuine quantum chaos with a positive Lyapunov exponent.

Consider the setup shown in Fig. 1. A quantum particle with mass $m$ moves freely between a rigid wall at $x=0$ and a mobile wall (a piston) with mass $M$ at $x=q$. Thus, the particle experiences a potential

$$v(x)=\begin{cases}\infty & \text{for} \quad x\leqslant 0 \text{ or } x\geqslant q\\ 0 & \text{for} \quad 0<x<q\end{cases} \tag{1}$$

The mobile wall moves in the potential ("nonlinear spring" in Fig. 1)

$$V(q)=V_0 q(q-Q)^2 \tag{2}$$

The force acting on the piston wall due to the nonlinear potential $V(q)$ is given by

$$F(q)=-\frac{\partial V}{\partial q}=-V_0(q-Q)(3q-Q) \tag{3}$$

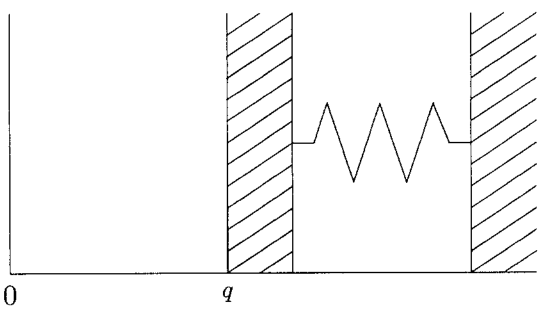


Fig. 1. Sketch of the quantum piston model.

In addition to $F$, the piston experiences a force due to the pressure of the quantum particle on the piston wall. For fixed $q$ the normalized eigenstates of the quantum particle are given by

$$\varphi_n(x; q) = (2/q)^{1/2} \sin(n\pi x/q), \qquad n = 1, 2, \ldots \tag{4}$$

For every piston position $q$ the wavefunction of the quantum particle can be expanded according to

$$\psi(x; q) = \sum_{n=1}^{\infty} A_n(q)\, \varphi_n(x; q) \tag{5}$$

This expansion inserted into the time-dependent Schrödinger equation yields

$$\begin{aligned}
i\dot{A}_n &= \sum_{k=1}^{\infty} D_{nk} A_k \\
D_{nk} &= \frac{1}{\hbar q^2} \varepsilon_n \delta_{nk} - i \frac{\dot{q}}{q} \mu_{nk} \\
\varepsilon_n &= \frac{\hbar^2 \pi^2}{2m} n^2 \\
\mu_{nk} &= \begin{cases} 0 & \text{for} \quad n = k \\ (-1)^{n+k} \dfrac{2nk}{n^2 - k^2} & \text{for} \quad n \neq k \end{cases}
\end{aligned} \tag{6}$$

Since $\mu_{nk}$ is antisymmetric, the coupling matrix $D$ is Hermitian and the total quantum probability

$$P = \sum_{n=1}^{\infty} |A_n|^2 \tag{7}$$

is conserved. The total kinetic energy of the quantum particle is given by

$$K(q) = \frac{1}{q^2} \sum_{n=1}^{\infty} \varepsilon_n |A_n|^2 \tag{8}$$

The kinetic energy acts like an additional potential for the piston. Therefore, the force acting on the piston due to the quantum particle is

$$G(q) = -\frac{\partial K}{\partial q} = \frac{2}{q^3} \sum_{n=1}^{\infty} \varepsilon_n \left[ |A_n|^2 + \mathrm{Re} \sum_{k=1}^{\infty} A_n^* \mu_{nk} A_k \right] \tag{9}$$

The equations of motion (6) for the quantum amplitudes can now be supplemented with the classical equations of motion for the piston wall. Introducing the piston momentum $p$, we have

$$\dot{q} = p/M; \qquad \dot{p} = F(q) + G(q) \tag{10}$$

Besides the quantum probability $P$, the total energy of the system

$$E = \frac{p^2}{2M} + V(q) + K(q) \tag{11}$$

is also a constant of the motion. In order to investigate the properties of the flow defined by (6) and (10), it is instructive to split the amplitudes $A$ into real and imaginary parts,

$$\begin{aligned} \dot{A}_n^{(r)} &= \frac{1}{\hbar q^2} \varepsilon_n A_n^{(i)} - \frac{\dot{q}}{q} \sum_{k=1}^{\infty} \mu_{nk} A_k^{(r)} \\ \dot{A}_n^{(i)} &= -\frac{1}{\hbar q^2} \varepsilon_n A_n^{(r)} - \frac{\dot{q}}{q} \sum_{k=1}^{\infty} \mu_{nk} A_k^{(i)} \end{aligned} \tag{12}$$

Because $\mu_{nn} = 0$, it is trivial to show that the flow defined by (10), (12) is divergence-free. This suggests that the set of equations (10), (12) can be derived from a Hamiltonian. Indeed, defining

$$X_n = \sqrt{2}\, Q A_n^{(r)}; \qquad Y_n = \sqrt{2} \frac{\hbar}{Q} A_n^{(i)} \tag{13}$$

the angular momentum

$$L = \sum_{nk} X_n \mu_{nk} Y_k \tag{14}$$

and the canonical momentum

$$w = p - L/q \tag{15}$$

we can derive the equations of motion for the piston model from the Hamiltonian

$$H = \frac{w^2}{2M} + \frac{1}{2q^2} \sum_{n=1}^{\infty} \varepsilon_n \left( \frac{1}{Q^2} X_n^2 + \frac{Q^2}{\hbar^2} Y_n^2 \right) + V(q) + \frac{Lw}{Mq} + \frac{L^2}{2Mq^2} \tag{16}$$

according to

$$\dot{q} = \frac{\partial H}{\partial w}; \qquad \dot{w} = -\frac{\partial H}{\partial q}; \qquad \dot{X}_n = \frac{\partial H}{\partial Y_n}; \qquad \dot{Y}_n = -\frac{\partial H}{\partial X_n} \tag{17}$$

Introducing dimensionless variables

$$\varepsilon = MQ^2E/\hbar^2; \qquad \eta = q/Q; \qquad \xi = wQ/\hbar$$
$$x = X/Q; \qquad y = QY/\hbar; \qquad \tau = \hbar t/MQ^2; \qquad l = L/\hbar \tag{18}$$

the mass ratio

$$\rho = M\pi^2/2m \tag{19}$$

and the dimensionless potential strength

$$v_0 = MV_0Q^5/\hbar^2 \tag{20}$$

we have that the Hamiltonian becomes

$$h = \frac{\xi^2}{2} + \frac{\rho}{2\eta^2}\sum_{n=1}^{\infty} n^2(x_n^2 + y_n^2) + v_0\eta(\eta-1)^2 + \frac{l\xi}{\eta} + \frac{l^2}{2\eta^2} \tag{21}$$

The equations of motion derive from

$$\dot{x}_n = \frac{\partial h}{\partial y_n}; \qquad \dot{y}_n = -\frac{\partial h}{\partial x_n}; \qquad \dot{\eta} = \frac{\partial h}{\partial \xi}; \qquad \dot{\xi} = -\frac{\partial h}{\partial \eta} \tag{22}$$

In order to gain more insight into the structure of this problem, we will consider two special cases.

(i) *One quantum level only.* In this case it is easy to show that the equations of motion are completely integrable.

(ii) *Two quantum levels.* This case leads to quantum chaos in the amplitudes $x_1$, $x_2$, $y_1$, and $y_2$. For this special case, the equations of motion are

$$\begin{aligned}
\dot{x}_1 &= \frac{\rho}{\eta^2}\, y_1 - \frac{\mu}{\eta}\left(\xi + \frac{l}{\eta}\right) x_2 \\
\dot{x}_2 &= 4\frac{\rho}{\eta^2}\, y_2 + \frac{\mu}{\eta}\left(\xi + \frac{l}{\eta}\right) x_1 \\
\dot{y}_1 &= -\frac{\rho}{\eta_2}\, x_1 - \frac{\mu}{\eta}\left(\xi + \frac{l}{\eta}\right) y_2 \\
\dot{y}_2 &= -4\frac{\rho}{\eta^2}\, x_2 + \frac{\mu}{\eta}\left(\xi + \frac{l}{\eta}\right) y_1 \\
\dot{\eta} &= \xi + \frac{l}{\eta} \\
\dot{\xi} &= \frac{\rho}{\eta^3}\sum_{n=1}^{2} n^2(x_n^2 + y_n^2) - v_0(\eta-1)(3\eta-1) + \frac{l\xi}{\eta^2} + \frac{l^2}{\eta^3}
\end{aligned} \tag{23}$$

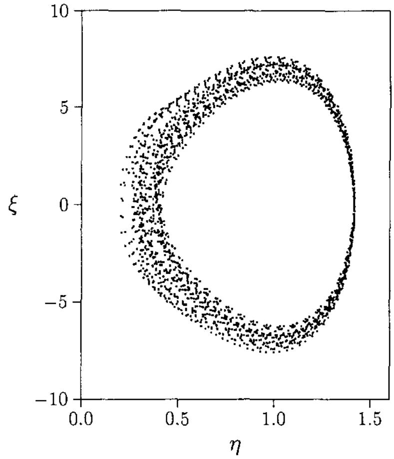


Fig. 2. Poincaré section in the $(\eta, \xi)$ plane of the quantum piston model.

with $\mu \equiv \mu_{12} = \frac{4}{3}$. The phase space is six-dimensional. But since $P = x_1^2 + x_2^2 + y_1^2 + y_2^2$ and $\varepsilon$ are conserved, the dimensionality is effectively reduced to four. Also, for the time being, we are not concerned with a global phase, which can be extracted from the set of quantum amplitudes. This reduces the dimension further to effectively three. A Poincaré surface of section with $y_2 = 0$ (see Fig. 2) reveals that there are no additional constants of the motion. This is an indication for chaos. Figure 3 shows the Euclidean distance $d$ (in six-dimensional phase space) of two trajectories started with $x_1^{(1)} = x_1^{(2)} = 1$, $x_2^{(1)} = x_2^{(2)} = 0$, $y_1^{(1)} = y_1^{(2)} = 0$, $y_2^{(1)} = y_2^{(2)} = 0$, $\eta^{(1)} = 0.2$, $\eta^{(2)} = \eta^{(1)} + \Delta$, $\xi^{(1)} = 0.1$, and $\xi^{(2)} = [2\varepsilon - \rho/\eta^{(2)^2} - 2v_0\eta^{(2)}(\eta^{(2)} - 1)^2]^{1/2}$. We chose $\Delta$ to be $\Delta = 10^{-8}$, $\rho = 1$, and $v_0 = 100$. Figure 3 clearly shows exponential separation, the hallmark of chaos in a bounded system. The exponential separation breaks only when at $\tau \approx 18$

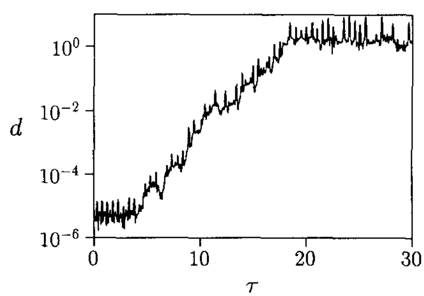


Fig. 3. Euclidean separation of two initially close trajectories as a function of time.

the size of the system is reached. The Lyapunov exponent which corresponds to the slope in Fig. 3 is $\lambda \approx 0.9$.

The ultimate proof of quantum chaos in this system, however, is exponential sensitivity in the quantum amplitudes. To show that this is indeed the case, we started two trajectories with $x_1^{(1)} = 1$, $x_2^{(1)} = 0$, $y_1^{(1)} = 0$, $y_2^{(1)} = 0$, $x_1^{(2)} = 1 - \Delta$, $x_2^{(2)} = 0$, $y_1^{(2)} = 2\Delta - \Delta^2$, $y_2^{(2)} = 0$, $\xi^{(1)} = \xi^{(2)} = 0.1$, $\eta^{(1)} = \eta^{(2)} = 0.2$. Figure 4 shows the Euclidean distance $d$ between the trajectories (1) and (2) in the four-dimensional "quantum" subspace $\{x_1, x_2, y_1, y_2\}$. An exponential growth of $d$ is clearly visible. We restricted ourselves to evaluation of $d$ in the four-dimensional quantum subspace to emphasize that exponential sensitivity can be defined for the quantum subsystem only. The quantum Lyapunov exponent corresponding to Fig. 4 is again $\lambda \approx 0.9$.

Thus we proved that a quantum system interacting self-consistently with a "wobbly" boundary condition exhibits chaos in the traditional sense of a positive Lyapunov exponent. Similar ideas were recently published in the context of nuclear collective motion.[9]

While it may be hard to set up an actual quantum system which is confined by a mobile macroscopic wall (an electron in a liquid helium bubble[10] may be a possibility), such systems are readily available in the acoustic context, where a sound field can act on its confining boundary and change it self-consistently. One may think about an actual acoustic experiment with a setup according to Fig. 1, a sound wave replacing the wave function of the quantum particle. Another possibility is sketched in Fig. 5. A hard sphere interacts with a sound wave in a rigid (stadium-shaped) enclosure. The sound wave imparts momentum to the sphere, which will start to move. This, on the other hand, will change the boundary condi-

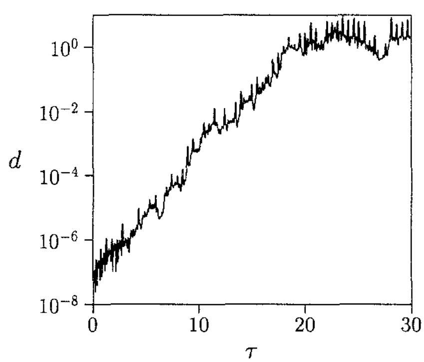


Fig. 4. Euclidean separation of initially close quantum amplitudes. The distance between amplitudes increases exponentially and allows one to define a quantum Lyapunov exponent.

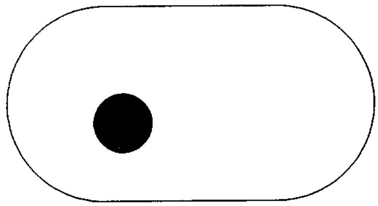

Fig. 5. Sketch of a system for the investigation of acoustic or quantum wave chaos. A sphere is enclosed in a stadium-shaped container and interacts self-consistently with a sound or matter wave trapped in the same container.

tions for the sound field, which has to adapt to the new position of the sphere. Based on our results for the quantum model studied above, we are confident of obtaining acoustic wave chaos for the system depicted in Fig. 5. Wave chaos, here, is to be understood in analogy to classical chaos. The time evolution of the wavefunction for the setup in Fig. 5 is expected to show exponential sensitivity to small changes in its initial configuration as well as a vanishing autocorrelation function $\langle \psi^*(\tau)\, \psi(\tau+\delta\tau) \rangle$ averaged on $\tau$ for $\delta\tau \to \infty$ (and generic initial conditions). Of course one may also solve the Schrödinger equation with Dirichlet boundary conditions for the system shown in Fig. 5. If the initial configuration of the matter wave is chosen such as to exert a net pressure on the sphere, the sphere would start moving and impart its resulting chaotic motion via the boundary conditions on the field distribution of the matter wave. Genuine quantum chaos in the usual sense of exponential sensitivity would result.

In conclusion, we mention that the piston model discussed above can also be interpreted as a classical one-dimensional oscillator coupled to a quantum degree of freedom. Since a one-dimensional oscillator is integrable, the results obtained in this paper show that despite the fact that quantum mechanics usually suppresses classical chaos[1, 3, 11, 12] here it is the catalyst which *enables* chaos to occur.

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